Dynamical systems with a Hamiltonian that is a function of momentum moduli: Pseudobilliards

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We consider a class of Hamiltonian dynamical systems with two degrees of freedom of the form: $H=c_1|p_1|+c_2|p_2|+U(x_1,x_2)$. Equations of motion for such systems can be easily integrated into successive time intervals; thus, their evolution can be found explicitly. On the other hand, these systems have a plethora of properties typical of nonintegrable Hamiltonian systems that are actively used in physics. This makes them quite good perspective models for a study of phenomena associated with such properties. As an example, a system with a quadratic potential is studied. [S1063-651X(97)01706-6]

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Dynamical systems that can be exactly analyzed always attract particular interest. We only mention the progress that has recently been made in the theory of exactly integrable systems [1] and the theory of billiard systems [2,3]. For the systems of the former class, the dynamics is simple and regular, whereas for those of the latter one, the dynamics is, in general, quite complex.

In this paper we introduce and study an important class of dynamical systems with two degrees of freedom with the Hamiltonian function of the form

$$H=c_1|p_1|+c_2|p_2|+U(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}^2 \tag{1}$$

($|\ |$ denotes a modulus, $c_1,c_2>0$ are constants). Below we show that such systems are, in some sense, an intermediate case between the two above classes. Moreover, qualitative behavior of their explicitly found ''pseudobilliard'' trajectories is rather similar to that of usual nonintegrable systems with two degrees of freedom.

Formally, Hamiltonian (1) describes the dynamics of two massless interacting particles. It corresponds to the following formal system of equations:

$$\dot{x}_1=\frac{\partial H}{\partial p_1}=c_1\ \mathrm{sgn}\ p_1,\quad \dot{x}_2=\frac{\partial H}{\partial p_2}=c_2\ \mathrm{sgn}\ p_2, \tag{2}$$

$$\dot{p}_1=-\frac{\partial H}{\partial x_1}=-\frac{\partial U}{\partial x_1},\quad \dot{p}_2=-\frac{\partial H}{\partial x_2}=-\frac{\partial U}{\partial x_2}. \tag{3}$$

(Here ''sgn'' is the standard signum function; the constants $c_1,c_2>0$ define possible values of the velocity projections $\dot{x}_1$ and $\dot{x}_2$.) These equations are canonical for Hamiltonian (1) everywhere except for the points where $p_1=0$ and/or $p_2=0$; at those points the dynamics is to be defined in its own right. Note that the dynamical system does not allow for the Lagrange description; in the case of one degree of freedom this statement is well known [4].

Let us show that the equations of motion (2) and (3) can be integrated into each finite time interval from the sequence $\{(T_n,T_{n+1}),\ n\in Z_+\}$. A right end point of each interval is defined as a moment when one of the momenta vanishes first or both vanish simultaneously (hence, the signs of both p_1 and p_2 inside each interval are constant). Let $\{x_1(0),p_1(0),x_2(0),p_2(0)\}$ be the initial data; then at $t>0$ by virtue of Eqs. (2) and (3) we get

$$x_j(t)=x_j(0)+c_jt\ \mathrm{sgn}\ p_j(0), \tag{4}$$

$$p_j(t)=p_j(0)-\int_0^t d\tau\frac{\partial U}{\partial x_j}(x_1(\tau),x_2(\tau)),\quad j=1,2. \tag{5}$$

Thus, as long as $p_1,p_2\neq 0$, the dynamics of the system is determined by solutions (4) and (5) of canonical equations (2) and (3). For any potential U the projection of the particular segment of the trajectory onto the plane $\{(x_1,x_2)\}$ is a segment of the straight line

$$x_2-x_2(0)=\frac{c_2}{c_1}\ \mathrm{sgn}\ \left(\frac{p_2(0)}{p_1(0)}\right)\ [x_1-x_1(0)]. \tag{6}$$

Note that the time dependence of x_j is specified by the first pair (4); substituting these expressions into Eq. (5) gives $p_j(t)$.

If one of the initial momenta $p_j(0)$ (or both) equals zero, then the canonical equations are not valid. In this case the Cauchy problem is naturally defined as a limit of the solution of system (2) and (3) as $t\to 0+$. In other words, since by virtue of Eq. (3)

$$p_j(\epsilon)=p_j(0)+\dot{p}_j(0)\epsilon+\cdots=p_j(0)-\left.\frac{\partial U}{\partial x_j}\right|_{t=0}\epsilon+\cdots \tag{7}$$

for small $\epsilon>0$, in expression (4) the following substitution is to be made:

$$\mathrm{sgn}\ p_j(0)\Rightarrow -\mathrm{sgn}\left.\frac{\partial U}{\partial x_j}\right|_{t=0}\qquad \text{if}\ p_j(0)=0,\quad j=1,2. \tag{8}$$

Such a definition of the Cauchy problem at the points of p_j vanishing guarantees H to be constant along the trajectory and, it should be noted, meets the spirit of the Fermat-

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Maupertuis principle (indeed, an analogous procedure is used to define the motion of a particle that has reached the boundary of a billiard [2]).

Analogously, if one of the momenta (or both) vanishes on a critical line of the potential $[\partial U/\partial x_1=0,\ \partial U/\partial x_2\neq 0$ or $\partial U/\partial x_2=0,\ \partial U/\partial x_1\neq 0]$ or at its critical point $[\partial U/\partial x_1=\partial U/\partial x_2=0]$, then the dynamics of the system is defined through an analysis of the motion in its vicinity as $t\to 0+$.

The evolution of the dynamical system is uniquely determined by expressions (4) and (5) until one of the momenta (or both simultaneously) vanishes. A corresponding moment T is, obviously, the least positive root of two (transcendental, in the general case) equations

$$\int_0^{t^{(j)}} d\tau \frac{\partial U}{\partial x_j}(x_1(\tau),x_2(\tau))=p_j(0), \quad j=1,2. \tag{9}$$

Let, for example, p_1 vanish at the moment $T_1=\min_{t^{(j)}>0}(\{t^{(1)}\},\{t^{(2)}\})$. Then for $t>T_1$ this momentum reverses sign and, by virtue of the first pair of equations (2), the projection of the trajectory onto the plane $\{(x_1,x_2)\}$ changes direction (turns through the angle that is equal to that between the lines $x_2=\pm c_2/c_1\ x_1$).

A simultaneous vanishing of p_1,p_2 corresponds to an equality of the least positive roots of both Eqs. (9). In such a situation both the momenta change sign (so, obviously, the trajectory returns to the state at the left end point of the time interval).

Further evolution is again defined using a limit of the solution of system (2) and (3) as $t\to T_1+0$ [in our example, where $p_1(T_1)=0$, this is reduced to the substitution (8) with $j=1$], and so forth.

Iterating this procedure, we build a sequence of time intervals $\{(T_n,T_{n+1}),\ n\in Z_+\}$. In projection onto the plane $\{(x_1,x_2)\}$ the corresponding trajectory is a broken curve which consists of straight segments of different length parallel either to the line $x_2=[c_2/c_1]x_1$ or to the line $x_2=-[c_2/c_1]x_1$. Thus, the motion of the particle in the potential $U(x_1,x_2)$ is somewhat similar to that in the rectangular billiard; however, in our model there are no fixed walls: the only restriction for the points of ''reflections'' (breaks) of the trajectory is to be inside the region determined by the condition $U(x_1,x_2)\leqslant H$. Our investigations have shown, in particular, that, depending on the initial conditions, the trajectories can demonstrate either regular or substantially chaotic behavior. In the first case they lie on some analogs of invariant tori, so that the ''break points'' (situated in one of the surfaces $p_1=0$ or $p_2=0$, which are suitable to use as Poincaré sections) lie on some invariant curves; in the second case these points fill the accessible region in a chaotic way.

Consider a simple model illustrating the properties of dynamics in systems of the class involved. Let the potential in Eq. (1) be defined by a quadratic form

$$U(x_1,x_2)=\tfrac{1}{2}(x_1^2+x_2^2)+\mu x_1x_2, \quad \mu\geqslant 0 \tag{10}$$

and $c_1=c_2=1$. The motion in this system is bounded in the interval $0\leqslant\mu<1$ which will be considered. Then within each interval the trajectory is determined by the following explicit expressions:

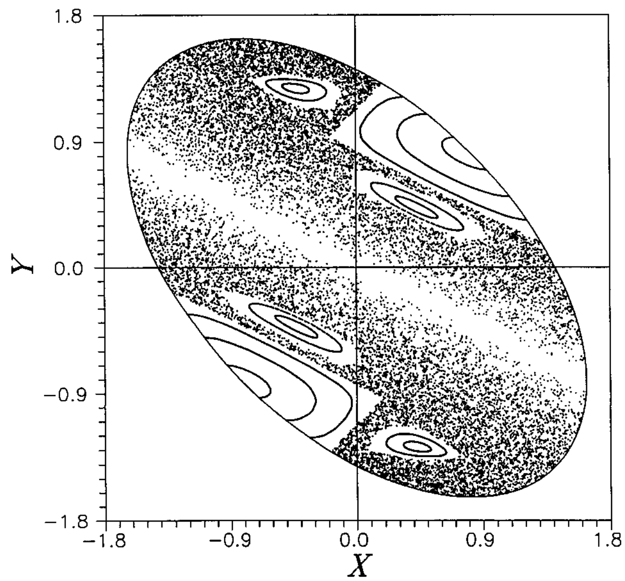

FIG. 1. The Poincaré mapping of the surface $p_2=0$ onto itself; $\mu=1/2$. The traces of several regular trajectories (they fill up the curves around the stationary points associated with stable periodic orbits) and the trace of one stochastic trajectory are shown. Hereafter: in projection onto the plane $\{(x_1,x_2)\}$, for $\mu=1/2$.

$$x_j(t)=x_j(t_n)+S_j(t-t_n), \quad j=1,2 \tag{11}$$

$$p_j(t)=p_j(t_n)-[x_j(t_n)+\mu x_{[2/j]}(t_n)](t-t_n) - \tfrac{1}{2}[S_j+\mu S_{[2/j]}](t-t_n)^2, \tag{12}$$

where

$$S_j\equiv\begin{cases}\operatorname{sgn} p_j(t_n), & p_j(t_n)\neq 0,\\ -\operatorname{sgn}[x_j(t_n)+\mu x_{[2/j]}(t_n)], & p_j(t_n)=0.\end{cases} \tag{13}$$

A typical graph of a Poincaré mapping of the surface $p_2=0$ into itself is given in Fig. 1. In Figs. 2 and 3 examples of regular and chaotic trajectories are shown.

Numerical calculation of the trajectories is reduced to solving the equations $p_1(t_{n+1})=0$ and $p_2(t_{n+1})=0$ [with the use of the above explicit expressions] followed by choosing the least positive root; as is seen from Eq. (12), for each segment of the trajectory one of these equations is linear and the other is quadratic. Investigation of system (1), (10) for μ varying in the interval $0\leqslant\mu<1$ shows that its dynamics has a number of properties similar to those of natural nonintegrable Hamiltonian systems with two degrees of freedom [5,6].

In the case $\mu=0$ the dynamics of the system is simple and typical for exactly integrable systems. As μ moves away from zero, there arise stochastic layers in the vicinity of destroyed separatrices related to unstable periodic orbits; they grow, intersect other stochastic layers, which generates a global chaotic motion (cf. [5]).

However, in the opposite limit case $\mu=1$ the system again admits a separation of variables (in one of them the

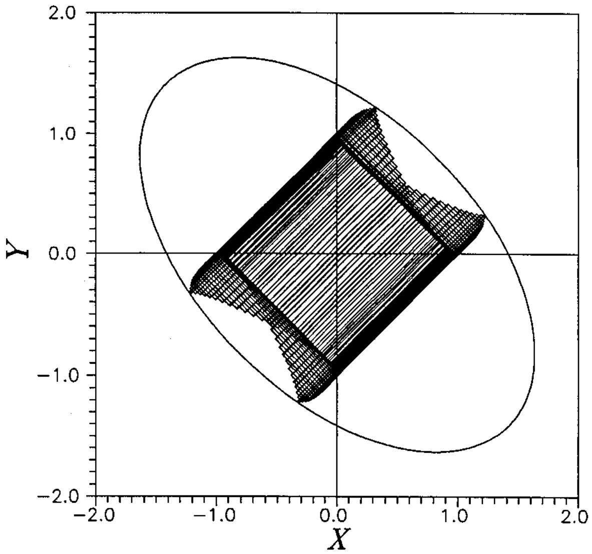

FIG. 2. A regular trajectory (fragment).

motion is free); the closer μ is to 1, the more regular is the dynamics: the chaotic region becomes more and more localized in the narrow layer around the line $x_2=-x_1$.

For $0\leqslant\mu<1$, the main structure of the phase space partition is determined by two of the simplest periodic orbits. One of them (label it "I_1") is projected onto the plane $\{(x_1,x_2)\}$ in a segment of the line $x_2=-x_1$ with the end points $(-a,a)$ and $(a,-a)$, $a=\sqrt{H/(1-\mu)}$; it is unstable for any $0<\mu<1$ (in Fig. 1 the traces of this orbit are the two points at which the line $x_2=-x_1$ intersects the ellipse that bounds a region accessible for trajectories at given H; these points are situated in the region of global chaos).

The second orbit "I_2" is projected in a segment of the line $x_2=x_1$ with the end points $(-b,-b)$ and (b,b), where $b=\sqrt{H/(1+\mu)}$; obviously, the period of motion is $4\sqrt{H/(1+\mu)}$.

The Poincaré mapping of the surface $p_j=0$ in the neighborhood of this orbit can be written in explicit form; it is area preserving and piecewise smooth. The vicinity of the stationary point associated with this orbit is divided into four sectors; for two of them the multiplicators equal ±1 for any μ, whereas for two other sectors they equal $F(\mu)\pm i\sqrt{1-F^2(\mu)}$, $F(\mu)\equiv(17\mu^2-14\mu+1)^2/(1+\mu)^4$, i.e., lie on the unit circle and take the value 1 for $\mu=0,1$ and the value -1 for $\mu=1/3$. This orbit is stable at almost all μ (except for a set of bifurcation points of measure 0). In Fig. 1 the traces of this orbit are the two points of intersection of the line $x_2=x_1$ with the ellipse; around these two points there are two large areas of conditionally regular motion.

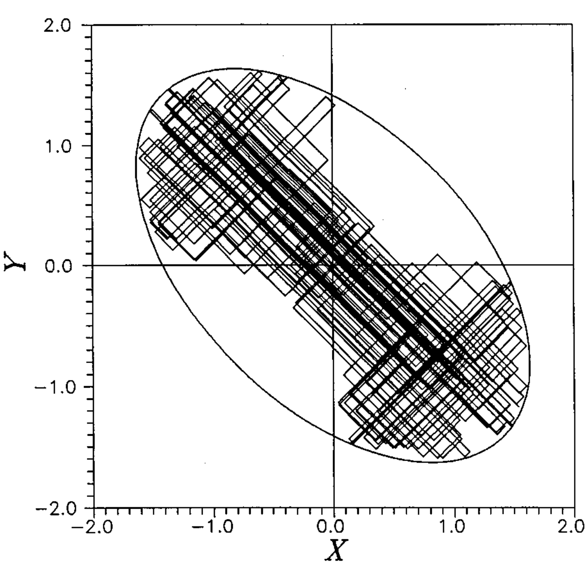

FIG. 3. A stochastic trajectory (fragment).

At $\mu=1/3$, when the multiplicators of this orbit pass through -1, a period doubling bifurcation occurs, just as in normal dynamical systems [5]: two pairs of periodic orbits branch off from the orbit I_2; they exist for $1/3<\mu<1$ and give rise to four smaller stability islands in Fig. 1. These orbits are also easily found in explicit form. For example, projections of the first pair onto the plane $\{(x_1,x_2)\}$ match and have the form of the "rectangular figure eight" oriented along the line $x_2=-x_1$; the movement along these orbits is performed for each orbit in its own direction; the "eight"'s size along the line $x_2=x_1$ is $D\equiv2\sqrt{2(1-\mu)H/(1-8\mu+23\mu^2)}$, and along the line $x_2=-x_1$ is $(3\mu-1)D/(1-\mu)$. Since the velocity of the motion is always constant and equals $\sqrt{2}$, the period of the orbit is easily calculated; at the bifurcation point it is $4\sqrt{3H}$, which is equal to the doubled period of the generating orbit I_2.

These periodic orbits are numerically found to be stable at μ lying in the interval from $\mu=1/3$ (their rise) to some bifurcation value $\mu\sim0.695$.

The second pair of periodic orbits arising at $\mu=1/3$ is unstable and together with the just described ones form typical resonance chains.

Up to $\mu\sim0.46$ the stochastic layer related to the above orbits is separated from the region of global chaos by nondestroyed invariant tori; at $\mu=1/2$, as is seen in Fig. 1, these two chaotic regions are already mixed.

Finally, let us describe an interesting phenomenon closely related to the intermittency [5]. It occurs when a trajectory enters the vicinity of one of two surfaces in the phase space defined by the conditions $\partial U/\partial x_j=0$, $p_j=0$ ($j=1$ or $j=2$) and situated (at $\mu>0$) in the region of global chaotic motion. Such a trajectory becomes trapped for some time in a narrow "channel" $-\delta<p_j<+\delta$ and moves along the line $\partial U/\partial x_j=0$, frequently changing the sign of the momentum p_j, whereas the sign of the other momentum is constant. The closer the trajectory is to this line at some moment, the more time it spends in its neighborhood. Thus, the critical potential lines are limit elements for such trajectories; in some sense, they are analogs of unstable (saddle) periodic orbits. Note, that the trajectories enter those neighborhoods comparatively seldom; this manifests itself in a low filling of the zone around the line $x_2=-\mu x_1$ in Fig. 1.

Thus, we have demonstrated that in systems of class (1) simple and complex periodic orbits, their stability and bifurcations can be studied analytically, so these systems are excellent models for development of the theory of dynamical systems.

To conclude, let us point out two evident generalizations of the proposed class. The first is inspired by the work of Zaslavsky and Sagdeev [7]. Let $(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_N)$ be a set of N nonidentity and nonparallel vectors, which form a ''hedgehog.'' Then for the dynamical system with the Hamiltonian

$$H = \sum_{n=1}^{N} |\vec{k}_n \vec{p}| + U(\vec{r}), \quad \vec{r}(x_1, x_2) \in \mathbb{R}^2 \tag{14}$$

the natural generalizations of the above results hold. In such a model a projection of the trajectory onto the plane $\{x_1, x_2\}$ is a broken line whose straight segments can only be parallel to N selected directions specified by the hedgehog.

Another, physically more important, generalization is related to introducing a Hamiltonian of the form

$$H = c_1(x_1, x_2)|p_1| + c_2(x_1, x_2)|p_2|. \tag{15}$$

In a one-dimensional case an analogous Hamiltonian governs, for example, a propagation of massless particles in inhomogeneous medium whose properties are described by the function $c(x)$. Note that, unlike systems of type (1) or (14), in this model the trajectories are broken lines formed by segments of curves.

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